From this it follows that in both inertial coordinates $\mathcal{K}$ and $\mathcal{K}'$ Newton's laws are identical, and thus *invariant* to transformations from one inertial frame to another.

The term "*special*" in the theory term 'special relativity' stems precisely from the fact that it refers *only* to inertial frames, and not to general coordinate systems, such as accelerated ones.

9.2 Minkowski space (MS)

Besides the postulate of invariance of physical processes between inertial frames (Section 9.1), there is a second one, the *constancy of the speed of light*: The speed of light $c = 3 \times 10^8\, m/s$ of a light signal measured relative to one inertial frame $\mathcal{K}$ is identical with the measurement of this light signal relative to a second inertial frame $\mathcal{K}'$ moving at constant speed.

As a consequence, we have to extend the description of an inertial frame from Section 9.1 as follows: Each point $\mathcal{P}$ (spatial vector $\vec{r}$) of an inertial coordinate system is equipped with a clock, time component t, which runs synchronously to each other, the so-called *proper time.* The coordinate system of SR thus consists of coordinate points $(t, \vec{r})$, which are called *events.* An event is therefore defined by a location and a time information to an inertial frame $\mathcal{K}(t, \vec{r})$. A four-dimensional flat space of events $\left(t, \vec{r}\left(x^1, x^2, x^3\right)\right)$ in an orthogonal coordinate system is called *Minkowski*[1] *Space.* When we speak of the Minkowski space in the present text, we use the short form MS or in modern terms called *spacetime*, shortened ST.

Remark 9.1
At the beginning of this chapter we speak mainly of *events* as coordinate points to emphasize the indissoluble connection of time and space in the Minkowski space of the SR. Only later in the chapter will we return to the usual term '*point*'.

■

The distance $(\triangle s)^2$ or *interval* between two events of an inertial frame $\mathcal{K}$ is

[1] Hermann Minkowski, 1864-1909

defined as follows:

$$(\triangle s)^2 = -c^2\,(\triangle t)^2 + (\triangle \vec{r})^2\,. \tag{9.2}$$

Definition 9.1
In order not always to carry the "ballast" of the large number $c = 3 \times 10^8\, m/s$ in the coordinates, the speed of light is set to 1, $c = 1$. Thus the time coordinate t gets the dimension of a length. We are already accustomed to this in astronomy, where distances from stars are given in time: the light years.

$$\begin{aligned} c &= 3 \times 10^8\, ms^{-1} = 1, \\ 1\, s &= 3 \times 10^8\, m, \\ 1\, m &= \frac{1}{3 \times 10^8}\, s. \end{aligned}$$

If in the further course of the text we give information on a velocity v, we speak of a dimensionless quantity v $\left[\frac{m}{s} = \frac{m}{s = 3\times10^8} = \frac{1}{3\times10^8}\right]$, which represents the proportion of the speed of light.

■

And so we adjust our equation for the interval of two events equation (9.2) with $c = 1$:

$$(\triangle s)^2 = -\,(\triangle t)^2 + (\triangle \vec{r})^2\,. \tag{9.3}$$

The physical measurement of a light signal emitted at $(t_1, \vec{r}_1)$ and received at $(t_2, \vec{r}_2)$ follows the condition

$$\begin{aligned} |\vec{r}_2 - \vec{r}_1| &= t_2 - t_1, \\ (\triangle \vec{r})^2 &= (\triangle t)^2\,, \\ 0 &= -\,(\triangle t)^2 + (\triangle \vec{r})^2\,. \end{aligned} \tag{9.4}$$

The result is remarkable: light signal events have a zero distance! These events are called *lightlike* or *null separated*. Since a light signal always propagates at the same speed, regardless of the inertial frame $\mathcal{K}$ or $\mathcal{K}'$, the same applies for $\mathcal{K}'$:

$$0 = -(\triangle t')^2 + (\triangle \vec{r}\,')^2. \tag{9.5}$$

It can be shown (see Box 9.1) that in general

$$(\triangle s)^2 = (\triangle s^{'})^2, \text{ or } ds^2 = ds^{'2}. \tag{9.6}$$

The difference value between two events, the spacetime interval, the distance between two points, is independent of the selected inertial frame and thus an invariant quantity. One changes the inertial coordinates without influencing the event distances; i.e. these intervals are a *property of the physical process*, but not of the coordinate system. *This invariance of the interval is a fundamental theorem of the SR.* The events with $ds^2 < 0$ are called *timelike separated*, while the events with $ds^2 > 0$ are called *spacelike separated*.

Box 9.1 invariance of the interval, equation (9.6)
The following explanation follows the reasoning of L.D.Landau, E.M.Lifschitz "Theoretische Physik", Vol.2.

Since the infinitesimal intervals ds^2 and $ds^{'2}$ are scalars and of the same order, the transformation $ds^2 = F\, ds^{'2}$ can only consist of a dimensionless transformation coefficient k: $ds^2 = k\, ds^{'2}$, which depends only on the relation between the inertial frames $\mathcal{K}$ and $\mathcal{K}^{'}$. $\mathcal{K}$ and $\mathcal{K}^{'}$ differ only in the relative velocity $\vec{v}$, (dimensionless, see Definition 9.1), they have to each other. Here the magnitude of $|\vec{v}|$ must be considered, because a directional dependence would violate the isotropy of space.

We consider three inertial frames $\mathcal{K}1, \mathcal{K}2$ and $\mathcal{K}3$. The relative velocity between $\mathcal{K}1$ and $\mathcal{K}2$ is v_{21}, and that between $\mathcal{K}2$ and $\mathcal{K}3$ is v_{32}. Then obviously the following relationship must apply between the transformation coefficients k

$$\underbrace{k\,(v_{31})}_{\mathcal{K}1\longrightarrow\mathcal{K}3} = \underbrace{k\,(v_{32})}_{\mathcal{K}2\longrightarrow\mathcal{K}3}\ \underbrace{k\,(v_{21})}_{\mathcal{K}1\longrightarrow\mathcal{K}2},$$

where v_{31} is the relative velocity between $\mathcal{K}1$ and $\mathcal{K}3$. The LHS depends on the angle that the speeds v_{21} and v_{32} include. The RHS has no reference to an angle. This contradiction is only resolved by the property that k is constant. And thus: $k = k^2$. The only non-trivial, real solution is: $k = 1$.

This proved the validity of the relationship from (9.6)

The canonical metric tensor of the MS is

$$\eta = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \ \eta_{ij} = \begin{cases} -1 & \text{for } i = j = 0 \\ 1 & \text{for } i = j = 1, 2, 3 \\ 0 & \text{for } i \neq j \end{cases} \tag{9.7}$$

The indices start with the number 0, representing the time component. The space components run as usual in geometry and vector calculus in ascending order over the natural numbers. And so we can express the line element ds^2 from (9.6) using the Minkowski metric as follows:

$$\begin{aligned} ds^2 &= \eta_{ij} dx^i dx^j \\ &= \eta_{00} \left(dx^0\right)^2 + \eta_{11} \left(dx^1\right)^2 + \eta_{22} \left(dx^2\right)^2 + \eta_{33} \left(dx^3\right)^2 \\ &= -dt^2 + \left(dx^1\right)^2 + \left(dx^2\right)^2 + \left(dx^3\right)^2, \\ &= -dt^2 + dx^2 + dy^2 + dz^2. \end{aligned} \tag{9.8}$$

Since ds^2 is a *frame invariant scalar quantity*, (9.6), it is called *proper time* for a time like event, $ds^2 < 0$,

$$d\tau^2 = -\eta_{ij} dx^i dx^j, \tag{9.9}$$

and *proper length* for a *space like* event, $ds^2 > 0$,

$$ds^2 = \eta_{ij} dx^i dx^j. \tag{9.10}$$

Remark 9.2
The proper time is not a constant, as the first impression from the name might suggest. Each point $\mathcal{P}(x^i)$ of the manifold MS has its own proper time $d\tau = \sqrt{-\eta_{ij} dx^i dx^j}$. $d\tau$ is a scalar and thus invariant to Lorentz transformations (see Box 9.3).

■

Based on equation (9.8), the question can be asked at which geometric locations are the event intervals ds^2 at equal distances from the origin of the inertial coordinates $\mathcal{K}$? For the geometrical answer we restrict ourselves to a 2-dimensional MS with $\eta = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, i.e. with the coordinates (t, x). Since we refer to the origin: $dt = t$, $dx = x$. We thus obtain three equations:

$$\begin{aligned} -t^2 + x^2 &= -{d_1}^2 < 0, \text{ timelike}, \\ -t^2 + x^2 &= 0, \text{ lightlike or null separated}, \\ -t^2 + x^2 &= {d_2}^2 > 0, \text{ spacelike}. \end{aligned} \tag{9.11}$$

The geometric location for constant timelike and spacelike intervals yields *hyperbolas*. Lightlike intervals are located on the straight lines $t = \pm x$. We enter these three position curves into a (t, x)-coordinate system and obtain the following Figure 9.1. In this diagram the lightlike events form a kind of dividing line between

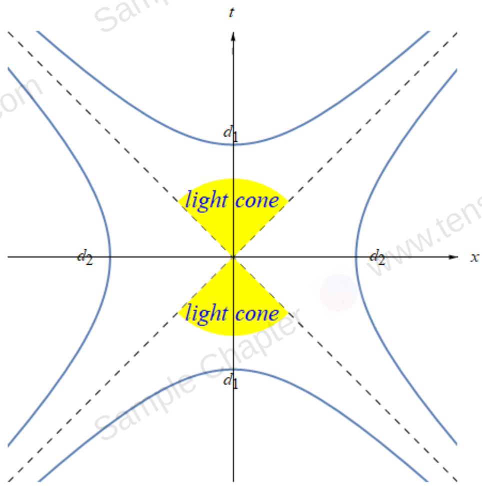

Figure 9.1: Invariant interval hyperbolas for the distances d_1 (timelike) and d_2 (spacelike). The dashed straight lines, asymptotes of the hyperbolas, denote the lightlike events.

the timelike and spacelike events. If one were to add the y-coordinate in Figure 9.1, one would obtain a cone surface, the so-called *light cone*, in place of the two lightlike straight lines. Since physical objects can only move at speeds below the speed of light, physical events are therefore only timelike events, and are therefore only located inside the light cone. The path of a particle in spacetime, i.e. the *sequence of events*, is called a world line.

The *spacetime diagram* in Figure 9.1 has a different geometric meaning than what we are used to from geometric drawings. The points of the shown hyperbolas have an equal distance from the origin, although they seem to move further and further away from the origin the further one walks along the hyperbola branch. It is important to make this mental change, because spacetime diagrams are a powerful tool to describe processes in MS. We will discuss this in more detail in Section 9.4. The fact that time is assigned its own coordinate axis is due to the clearer graphic representation in spacetime diagrams. Actually, *all points are inherently assigned their own clock*, which run synchronously with each other. We want to represent the hyperbolic branch d_1 from Figure 9.1 in the way just described, that each point has an *inherent clock* and the same event is observed from different points (Figure 9.2).

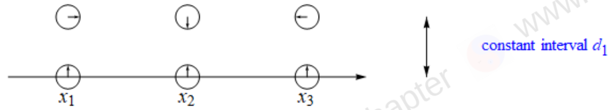

Figure 9.2: Hyperbola d_1 from Figure 9.1 in an alternative representation of clocks attached to positions.

In a two-dimensional Euclidean space with the metric $g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ we would obtain a *circle* for the location of the points which are at the same distance from the origin. In MS (2-dimensional), *hyperbolas* indicate equal distances (see Figure 9.3).

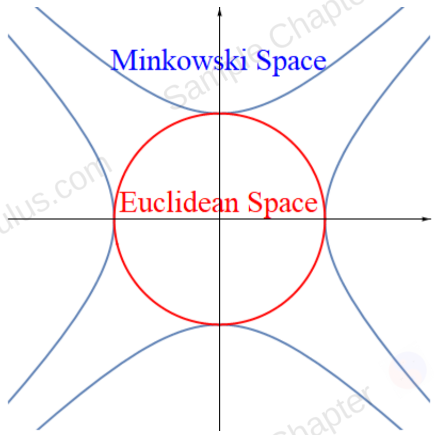

Figure 9.3: Pictorial representation of constant intervals of magnitude 1 for the Euclidean space and the Minkowski space of special relativity..

Box 9.2 *philosophical perspective*
Figure 9.3 shows the change in world view in the 19th century. Until the 19th century, the 'Weltanschauung' of mankind was a closed one. It was finite and "traceable". When physics gained new insights into the micro- and macrocosm at the transition to the 20th century, the world view changed dramatically. A "closed circle" became an "open hyperbole". Man was thrown out of his cosy warm nest into the infinitely and cold universe.