7.8 Symmetry properties of the Riemann tensor

The properties of a tensor are coordinates independent. In order to investigate the *symmetry properties* of the Riemann tensor **R**, we do so in a local inertial frame, Section 7.2, because there the expression is simplified due to $\Gamma^k_{ij} = 0$, conditions (7.6). And in this way we simplify the calculation work. With equation (7.14) we get:

$$R^k{}_{sji} = \partial_j \Gamma^k_{is} - \partial_i \Gamma^k_{js}.$$

We obtain the derivative of the connection coefficients Γ from the Levi-Civita connection formula, equation (6.25), noting that the derivative of the first factor g^{kt} is zero due to the constraints of a local inertial frame (7.6):

$$\partial_j \Gamma^k_{is} = \frac{1}{2} g^{kt} \left(\partial_j \partial_i g_{st} + \partial_j \partial_s g_{ti} - \partial_j \partial_t g_{is} \right),$$
$$\partial_i \Gamma^k_{js} = \frac{1}{2} g^{kt} \left(\partial_i \partial_j g_{st} + \partial_i \partial_s g_{tj} - \partial_i \partial_t g_{js} \right).$$

And therefore

$$\begin{aligned} R^k{}_{sji} &= \partial_j \Gamma^k_{is} - \partial_i \Gamma^k_{js} \\ &= \frac{1}{2} g^{kt} \left(\partial_j \partial_i g_{st} + \partial_j \partial_s g_{ti} - \partial_j \partial_t g_{is} - \partial_i \partial_j g_{st} - \partial_i \partial_s g_{tj} + \partial_i \partial_t g_{js} \right). \end{aligned}$$

Because of $\partial_d \partial_c g_{ab} = \partial_c \partial_d g_{ab}$ we get:

$$R^k{}_{sji} = \frac{1}{2} g^{kt} \left(\partial_j \partial_s g_{ti} - \partial_j \partial_t g_{is} - \partial_i \partial_s g_{tj} + \partial_i \partial_t g_{js} \right).$$

We multiply both sides by $g_{lk}, g_{lk} g^{kt} = \delta^t_l$, subsequent $l \longrightarrow k$, and obtain:

$$R_{ksji} = \frac{1}{2} \left(\partial_j \partial_s g_{ki} - \partial_j \partial_k g_{is} - \partial_i \partial_s g_{kj} + \partial_i \partial_k g_{js} \right). \tag{7.15}$$

Remark 7.4
The expression (7.15) for the $(0,4)$ low-index Riemann tensor was derived assuming an inertial frame. It is used exclusively to investigate the symmetry properties of the Riemann tensor, because the symmetry properties of a tensor are independent of the coordinate system. To determine the components of the $(1,3)$ Riemann tensor (7.14) an index raising by means of the metric is necessary. See the further explanations and Remark 7.5.

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The following four symmetry properties of the Riemann tensor can be read very clearly from equation (7.15):

$$\begin{aligned} \text{index 1-2 swap}: R_{ksji} &= -R_{skji}, \\ \text{index 3-4 swap}: R_{ksji} &= -R_{ksij}, \\ \text{index (12)-(34) swap}: R_{ksji} &= R_{jiks}, \\ \text{Bianchi identity}: R_{ksji} &+ R_{kjis} + R_{kisj} = 0. \end{aligned} \tag{7.16}$$

Box 7.5
In a 2-dimensional space there are $2 \times 2 \times 2 \times 2 = 16$ possible different components of the Riemann tensor. However, due to symmetry properties, as we analyse, they will be reduced to 4 non-zero low index components, but reduced to only *one independent* component.

$$\begin{aligned} \text{1-2 swap}: & R_{11xx} = -R_{11xx} = 0,\ R_{22xx} = -R_{22xx} = 0, \\ \text{3-4 swap}: & R_{xx11} = -R_{xx11} = 0,\ R_{xx22} = -R_{xx22} = 0. \end{aligned}$$

Four non-zero components remain

$$R_{1212} = a, \underbrace{\Longrightarrow}_{\text{3-4 swap}} R_{1221} = -a, \underbrace{\Longrightarrow}_{\text{1-2 swap}} R_{2121} = a, \underbrace{\Longrightarrow}_{\text{3-4 swap}} R_{2112} = -a,$$

with the scalar value $\pm a$.

Example 7.2
We want to determine the component $R^r{}_{\theta r\theta}$ of the Riemann tensor in polar coordinates for Euclidean plane. Of course we expect the components of $\mathbf{R}$ to be zero, because in a flat manifold, independent of the selected coordinate frame, a vector can be shifted parallel on any closed path and brought back to coincidence.

We use the results of Box 6.3 where we have already determined the connection coefficients for polar coordinates. With (7.14) we obtain:

$$\begin{aligned} R^r{}_{\theta r\theta} &= \partial_r \Gamma^r_{\theta\theta} - \partial_\theta \Gamma^r_{r\theta} + \Gamma^r_{rl}\Gamma^l_{\theta\theta} - \Gamma^r_{\theta l}\Gamma^l_{r\theta} \\ &= \partial_r(-r) - \partial_\theta(0) + \underbrace{\Gamma^r_{rr}}_{=0}\Gamma^r_{\theta\theta} + \underbrace{\Gamma^r_{r\theta}}_{=0}\Gamma^\theta_{\theta\theta} - \underbrace{\Gamma^r_{\theta r}}_{=0}\Gamma^r_{r\theta} - \underbrace{\Gamma^r_{\theta\theta}}_{=-r}\underbrace{\Gamma^\theta_{r\theta}}_{=\frac{1}{r}} \\ &= -1 - (-1) = 0. \end{aligned}$$

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