5.2 Covector field or gradient

The differential d applied to a scalar field $f(x, y, z)$ yields by means of the chain rule:

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz.$$

However, if we let the differential operator $\tilde{\mathrm{d}}(\ )$ act on this scalar field $f(x, y, z)$ so the result is a *covector field* or *one-form*:

$$\tilde{\mathrm{d}}(f) = \tilde{\mathrm{d}}f = \frac{\partial f}{\partial x}\tilde{e}^x + \frac{\partial f}{\partial y}\tilde{e}^y + \frac{\partial f}{\partial z}\tilde{e}^z = \frac{\partial f}{\partial x^i}\tilde{e}^i \tag{5.3}$$

For $\frac{\partial f}{\partial x^i}$ we write the short form, see equation (1.5), $\partial_i f$, and thus get

$$\tilde{\mathrm{d}}f = (\partial_i f)\tilde{e}^i, \tag{5.4}$$

which is a covector (one-form) and therefore maps a vector to a real number. His components are arranged as a row vector:

$$\tilde{\mathrm{d}}f = (\partial_i f)\tilde{e}^i \equiv \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right). \tag{5.5}$$

This one-form or covector is called *gradient* of f.

Usually, in the three-dimensional vector calculus, the gradient is introduced as a vector. We'll talk about the difference between *covector gradient* and *vector gradient* later. The following Figure 5.2 shows a scalar field (the values increase from the blue area to the red area) and the associated gradient, the covector field. In the right picture of Figure 5.2 you can see the level set curves, the curves of constant value. The small arrows on the curves point in the direction of ascending values.

The components of a covector gradient $(\tilde{\mathrm{d}}f)_i$ transform just like the components of a one-form, see equation (2.12):

$$(\tilde{\mathrm{d}}f)_{j'} = (\partial_i f)\, F^i_{\ j'}, \quad F^i_{\ j'} = \frac{\partial x^i}{\partial x^{j'}}. \tag{5.6}$$

The *directional derivative*, the rate of change of a scalar field f along the direction (or directional velocity) $\vec{v}$ is defined by:

$$\begin{aligned} \tilde{\mathrm{d}}f(\vec{v}) &= (\partial_i f)\tilde{e}^i(v^j\vec{e}_j) = (\partial_i f)v^j\tilde{e}^i(\vec{e}_j) \\ &= (\partial_i f)v^j\delta^i_j = (\partial_i f)v^i. \end{aligned} \tag{5.7}$$

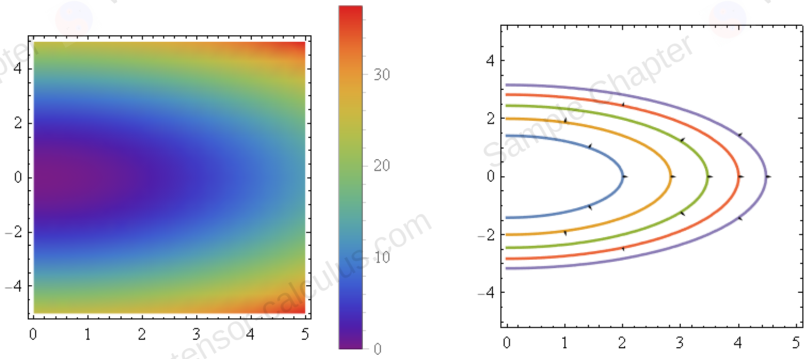

Figure 5.2: Scalar field and associated covector (one-form) gradient.

If we let the one-form gradient $\tilde{\mathrm{d}}f(\ )$ act on a displacement vector $d\vec{s}$, we get the following known differential (directional derivative):

$$\tilde{\mathrm{d}}f(d\vec{s}) = (\partial_i f)\tilde{e}^i(dx^j\vec{e}_j) = (\partial_i f)dx^j\delta^i_j = (\partial_i f)dx^i = df.$$

The gradient $\tilde{\mathrm{d}}f$ of the scalar field f lives in dual space $\mathcal{V}^\star$. If we look for its equivalent from the vector space $\mathcal{V}$, the *vector gradient* $\vec{\mathrm{d}}f$, we obtain this with the help of the dual metric tensor $\boldsymbol{g}$, equation (4.43):

$$\vec{\mathrm{d}}f = \boldsymbol{g}\left(\tilde{\mathrm{d}}f;\ \right). \tag{5.8}$$

$\vec{\mathrm{d}}f$ is a *vector field* (contravariant components), unlike $\tilde{\mathrm{d}}f$, which is a covector field (covariant components). For $\vec{d}$ one often writes the 'del' symbol $\nabla \equiv \partial^i(\ )\vec{e}_i$:

$$\vec{d}f = \nabla f = \underbrace{\partial^i f}_{\text{contravariant}} \vec{e}_i. \tag{5.9}$$

And for the components we get:

$$\begin{aligned} \partial^j f &= g^{ji}(\tilde{\mathrm{d}}f)_i, \\ \underbrace{\partial^j f}_{\text{contravariant}} &= g^{ji} \underbrace{\partial_i f}_{\text{covariant}} . \end{aligned} \tag{5.10}$$

The following simple example illustrates the difference between the components of covector gradient and vector gradient.

Example 5.1
For polar coordinates, the covector gradient components are:

$$(\partial_r f,\, \partial_\theta f)\,.$$

The polar metric tensor has the components, Section 4.3:

$$g_{rr} = 1,\; g_{r\theta} = g_{\theta r} = 0,\; g_{\theta\theta} = r^2.$$

And so we get for the dual metric tensor:

$$g^{rr} = 1,\; g^{r\theta} = g^{\theta r} = 0,\; g^{\theta\theta} = \frac{1}{r^2}.$$

The components of the vector gradient are thus:

$$\nabla f = \left(\partial^r f, \frac{1}{r^2}\,\partial^\theta f\right). \tag{5.11}$$

And the vectorial representation (with coordinate basis!) is:

$$\nabla f = \partial^r f\,\vec{e}_r + \frac{1}{r^2}\,\partial^\theta f\,\vec{e}_\theta.$$

If we introduce a normalised basis $\{\vec{e}_r = \hat{e}_r, \vec{e}_\theta = r\hat{e}_\theta\}$, we recognise the familiar formula for a vector gradient for polar coordinates from vector calculus

$$\nabla f = \partial^r f\,\vec{e}_r + \frac{1}{r^2}\,\partial^\theta f\,\vec{e}_\theta = \partial^r f\,\hat{e}_r + \frac{1}{r^2}\,\partial^\theta f\;(r\hat{e}_\theta) = \partial^{\hat{r}} f\,\hat{e}_r + \frac{1}{r}\,\partial^{\hat{\theta}} f\,\hat{e}_\theta,$$

with $\left(\partial^{\hat{r}} f, \partial^{\hat{\theta}} f\right)$ as contravariant components of the vector gradient with orthonormal basis.

In this way, the vector gradient, or its components, of a scalar field f can be determined very quickly by working with the partial derivatives of the scalar field $\partial_i f$ (covariant components) and the inverse of the metric tensor $\mathbf{g}$ to get the contravariant components $\partial^j f$ of the vector gradient.

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Box 5.1
The figural representation of the vector gradient ∇f is a directional vector that is perpendicular to a constant level line and points in the direction of increasing values. In this definition, we got to know the gradient in vector calculus.
We want to prove this with the tools of the past chapters. We start with

equation (5.8):

$$\nabla f = \boldsymbol{g}\left(\tilde{\mathrm{d}}f;\ \right).$$

Let $\vec{v}$ be a vector of $\mathcal{V}$. With Definition 4.1 and equation (4.41) we get:

$$\nabla f \cdot \vec{v} = \mathbf{g}\left(\,;\nabla f, \vec{v}\right) = \tilde{\mathrm{d}}f(\vec{v}).$$

$\tilde{\mathrm{d}}f(\vec{v})$ is according to equation (5.7) the rate of change of f along $\vec{v}$.
Now suppose that $\vec{t}$ is a tangent vector along a contour line. The rate of change along a contour is by definition zero. And thus:

$$\nabla f \cdot \vec{t} = 0.$$

Therefore the vector gradient ∇f is orthogonal to a contour line of f.