Chapter 4

Metric Tensors

4.1 Introduction

One of the most important tensors of differential geometry is the metric tensor $\mathbf{g}$. His name already describes one of his 'tasks', namely to determine the magnitude of a vector or a displacement. Just as the geometric space is made up of two spaces (vector space $\mathcal{V}$ and dual space $\mathcal{V}^\star$) that are interwoven, so there is also a second metric tensor, which, among other things, can determine the magnitude of a one-form. We denote it in this text the *dual metric tensor* $\boldsymbol{g}$ because it is capable of processing dual-space one-forms. The metric tensor $\mathbf{g}$ 'lives' in the dual space $\mathcal{V}^\star$, and the dual metric tensor $\boldsymbol{g}$ 'lives' in the vector space $\mathcal{V}$. The Figure 4.1 is an attempt to depict these spatial formations.

The interweaving of vector-space $\mathcal{V}$ and dual space $\mathcal{V}^\star$, of metric tensor $\mathbf{g}$ and dual metric tensor $\boldsymbol{g}$ is further illustrated by the fact that the components of $\mathbf{g} \in \mathcal{V}^\star$ are calculated from the inner product of the basis vectors $\{\vec{e}_i\}$ of the vector space $\mathcal{V}$, Section 4.3.

4.2 Inner product

Up to now we have only created a scalar value from a covector and vector: $\tilde{p}(\vec{v}) \rightarrow \mathbb{R}$, or $\vec{v}(\tilde{p}) \rightarrow \mathbb{R}$. To produce a scalar value from two vectors a so-called *inner product* is defined: $\vec{v} \cdot \vec{w} \rightarrow \mathbb{R}$. The vector space $\mathcal{V}$ in which an inner product is defined is called *inner product space*. The following properties must be fulfilled:

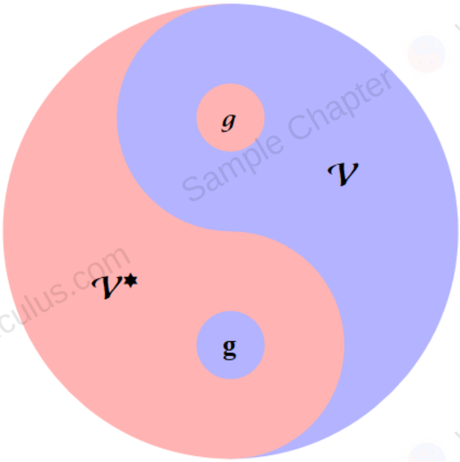

Figure 4.1: A symbol of the interweaving of vector space $\mathcal{V}$ and dual space $\mathcal{V}^\star$ with the metric tensor $\mathbf{g}$ and $\boldsymbol{g}$.

1. Bilinearity:

$$\begin{aligned}
&(a\vec{v}_1 + b\vec{v}_2)\cdot\vec{w} = a(\vec{v}_1\cdot\vec{w}) + b(\vec{v}_2\cdot\vec{w}),\\
&\vec{v}\cdot(c\vec{w}_1 + d\vec{w}_2) = c(\vec{v}\cdot\vec{w}_1) + d(\vec{v}\cdot\vec{w}_2)),\\
&\text{for all } \vec{v},\vec{v}_1,\vec{v}_2,\vec{w},\vec{w}_1,\vec{w}_2 \in \mathcal{V} \text{ and } a,b,c,d \in \mathbb{R}.
\end{aligned} \tag{4.1}$$

2. Symmetry:

$$\vec{v}\cdot\vec{w} = \vec{w}\cdot\vec{v},\ \text{for all } \vec{v},\ \vec{w} \in \mathcal{V}. \tag{4.2}$$

3. Nondegeneracy:

$$\vec{v}\cdot\vec{x} = 0, \text{for all } \vec{v}\in\mathcal{V} \Rightarrow \vec{x} = \vec{0}. \tag{4.3}$$

We consider in this textbook only symmetrical nondegenerate inner products. For a coordinate basis $\{\vec{e}_i\}$ applies to the inner product:

$$\vec{e}_i\cdot\vec{e}_j = \begin{cases}\mathbb{R} & i = k,\\ 0 & i \neq k.\end{cases} \tag{4.4}$$

4.3 Definition and components of g

The *metric tensor* **g** is defined by the inner product of two vectors. A inner product of two vectors yields a real number ($\mathcal{V}$ over $\mathbb{R}$). Therefore, the metric tensor **g** is an (0, 2)-tensor, $\mathbf{g}(,) \in \mathcal{V}^\star \otimes \mathcal{V}^\star$, which processes *two vectors as arguments* of any basis into a number. This numerical value is independent of the chosen vector basis. From the above definition follows immediately the linearity of the metric tensor in each of its arguments:

$$\mathbf{g}(a\vec{u} + b\vec{v}, \vec{w}) = (a\vec{u} + b\vec{v}) \cdot \vec{w} = a\vec{u} \cdot \vec{w} + b\vec{v} \cdot \vec{w} = a\,\mathbf{g}(\vec{u}, \vec{w}) + b\,\mathbf{g}(\vec{v}, \vec{w}). \tag{4.5}$$

And in the same way for the second argument. The metric tensor $\mathbf{g}(\vec{v}, \vec{w})$ is is thus a *bilinear form.*

Definition 4.1
The metric tensor **g** is defined by the *inner product* (dot product or scalar product) of two vectors:

$$\mathbf{g}(\vec{v}, \vec{w}) = \vec{v} \cdot \vec{w} = (v^i \vec{e}_i) \cdot (w^j \vec{e}_j) = (\vec{e}_i \cdot \vec{e}_j)\, v^i\, w^j.$$

The component representation of **g** is

$$\mathbf{g}(\;,\;) = g_{ij}\, \tilde{e}^i \otimes \tilde{e}^j(\;,\;) = g_{ij}\, \tilde{e}^i \tilde{e}^j(\;,\;).$$

The components of a tensor are determined by processing basis vectors. And thus we obtain:

$$\begin{aligned} \mathbf{g}(\vec{e}_m, \vec{e}_n) &= \vec{e}_m \cdot \vec{e}_n, \\ \mathbf{g}(\vec{e}_m, \vec{e}_n) &= g_{ij}\, \tilde{e}^i \tilde{e}^j(\vec{e}_m, \vec{e}_n) = g_{ij} \delta^i_m \delta^j_n = g_{mn}, \\ \Rightarrow g_{mn} &= \vec{e}_m \cdot \vec{e}_n. \end{aligned} \tag{4.6}$$

■

For the inner product of two vectors we get:

$$\vec{v} \cdot \vec{w} = g_{ij}\, v^i\, w^j. \tag{4.7}$$

Remark 4.1
Since basic vectors are generally location-dependent in direction and magnitude (polar coordinates, spherical coordinates, etc.), the inner product must always be calculated based on Cartesian coordinates.

■

Example 4.1
We determine the components of the metric tensor for *polar coordinates.* With equation (1.11) we get:

$$\begin{aligned}
\vec{e}_r \cdot \vec{e}_r &= (\cos\theta\,\vec{e}_x + \sin\theta\,\vec{e}_y)\cdot(\cos\theta\,\vec{e}_x + \sin\theta\,\vec{e}_y) = 1,\\
\vec{e}_r \cdot \vec{e}_\theta &= (\cos\theta\,\vec{e}_x + \sin\theta\,\vec{e}_y)\cdot(-r\sin\theta\,\vec{e}_x + r\cos\theta\,\vec{e}_y) = 0,\\
\vec{e}_\theta \cdot \vec{e}_r &= (-r\sin\theta\,\vec{e}_x + r\cos\theta\,\vec{e}_y)\cdot(\cos\theta\,\vec{e}_x + \sin\theta\,\vec{e}_y) = 0,\\
\vec{e}_\theta \cdot \vec{e}_r &= (-r\sin\theta\,\vec{e}_x + r\cos\theta\,\vec{e}_y)\cdot(-r\sin\theta\,\vec{e}_x + r\cos\theta\,\vec{e}_y) = r^2.
\end{aligned}$$

The components of the metric tensor for polar coordinates $\{r,\theta\}$ are thus:

$$g_{rr} = 1,\ g_{r\theta} = g_{\theta r} = 0,\ g_{\theta\theta} = r^2. \tag{4.8}$$

■

Since the inner product is symmetric,$\mathcal{V}$ over $\mathbb{R}$, $\vec{v}\cdot\vec{w} = \vec{w}\cdot\vec{v}$, the metric tensor is also *symmetric in its arguments.* Another argument for the symmetry of the metric tensor is as follows. In Cartesian coordinates:

$$g_{ij} = \vec{e}_i \cdot \vec{e}_j = \delta_{ij} = \delta_{ji} = g_{ji}. \tag{4.9}$$

g_{ij} is therefore symmetrical in Cartesian coordinates, and thus symmetrical in all other coordinate systems, because the properties of a tensor are frame invariant (Definition 3.1). Therefore

$$\mathbf{g}(\vec{v},\vec{w}) = \mathbf{g}(\vec{w},\vec{v}),\ g_{ij} = g_{ji}. \tag{4.10}$$

We now analyse the transformation behaviour of the metric components g_{ij}. With equation (4.6) and (1.13) applies

$$g_{i'j'} = \vec{e}_{i'} \cdot \vec{e}_{j'} = (\vec{e}_i F^i_{i'})\cdot(\vec{e}_j F^j_{j'}) = (\vec{e}_i\cdot\vec{e}_j)F^i_{i'}F^j_{j'} = g_{ij}F^i_{i'}F^j_{j'}. \tag{4.11}$$

If we write the (0,2) metric tensor g_{ij} in matrix form $(g) \equiv (g^i_j)$, in order to be able to continue to calculate with matrices, we must keep to the matrix multiplication rules (see Remark 1.5), and adjust the transformation matrix F accordingly. And thus equation (4.11) is written in matrix notation:

$$\begin{aligned} \left(g^{i'}_{j'}\right) &= \left(F^{i}_{i'}\right)^{T} (g^{i}_{j})\, F^{j}_{j'}, \\ \left(g^{'}\right) &= F^{T} (g)\, F. \end{aligned} \tag{4.12}$$

Example 4.2
Using equation (4.12) we want to determine the metric matrix $\left(g^{'}\right)$ for polar coordinates. In Box 1.3 we have already determined the transformation matrix $F = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$ for polar coordinates and with $(g) = \delta^{i}_{j}$ we get

$$\left(g^{'}\right)_{\text{polar}} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}^{T} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & r^{2} \end{pmatrix}. \tag{4.13}$$

■

Definition 4.2
The norm of a vector $\parallel \vec{v} \parallel$ is defined by the inner product:

$$\parallel \vec{v} \parallel^{2} = \vec{v} \cdot \vec{v} = \mathbf{g}(\vec{v}, \vec{v}) = g_{ij} v^{i} v^{j}. \tag{4.14}$$

■

4.4 Signature of the metric

The metric tensor $\mathbf{g}$ is symmetrical in its indices, Section 4.3 and equation (4.10). According to the spectral theorem, any symmetrical matrix can be diagonalized, has real eigenvalues Λ and and orthogonal eigenvectors Q. Thus applies:

$$\left(g^{'}_{\text{diagonal}}\right) = \Lambda = Q^{T}\,(g)\,Q. \tag{4.15}$$